What is an Automorphic Representation?

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- The main goal of this talk will be to make sense of everything highlighted in orange.

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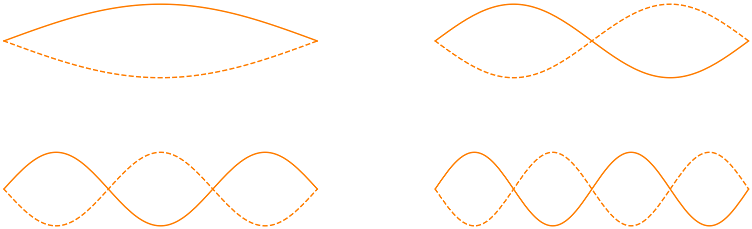

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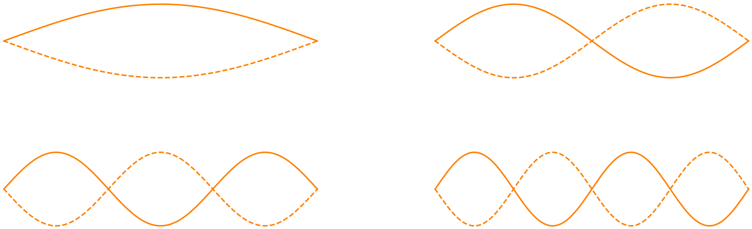

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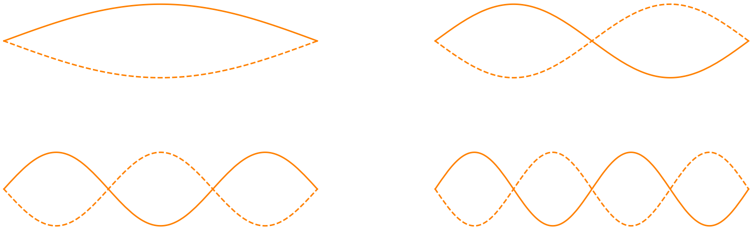

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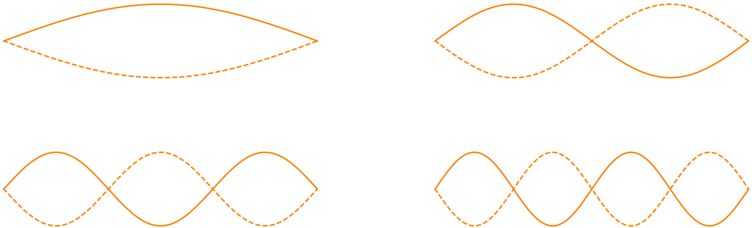

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- Fundamental modes encode the counting numbers!

More Complicated Objects?

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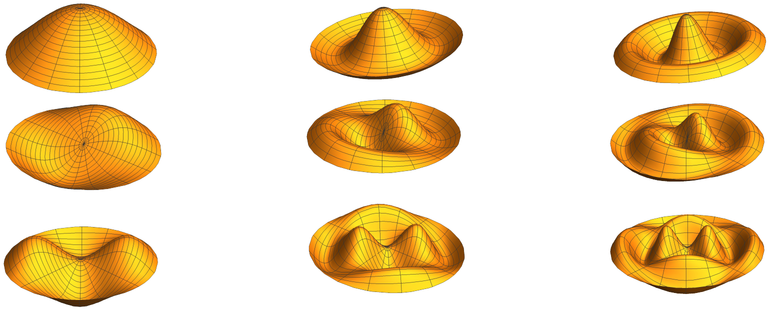

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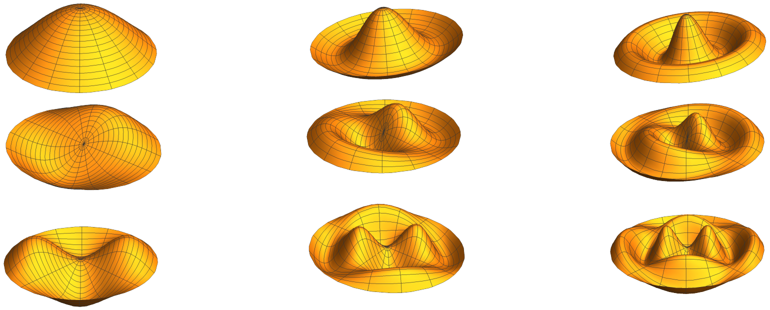

Frequencies are complicated (Bessel zeroes), different qualitative types from rotational symmetry ("weight" or "K-types")

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- Automorphic Representations are ones on *very specific objects*
- These will be abstract and physically impossible—math not physics!
- We will build up the objects through examples/analogies of increasing complexity

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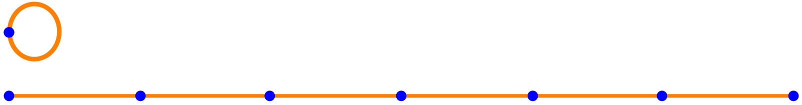

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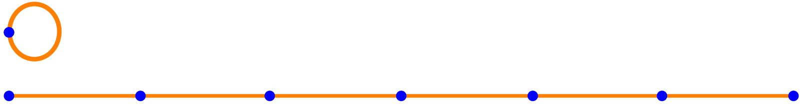

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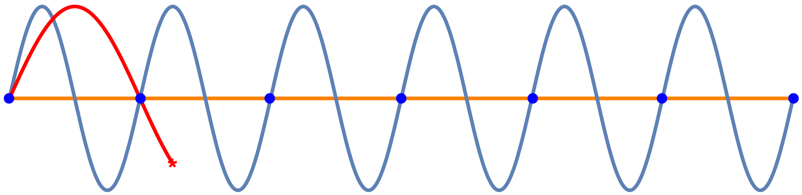

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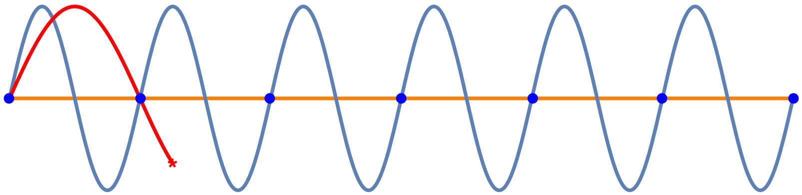

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- Still get one for each counting number

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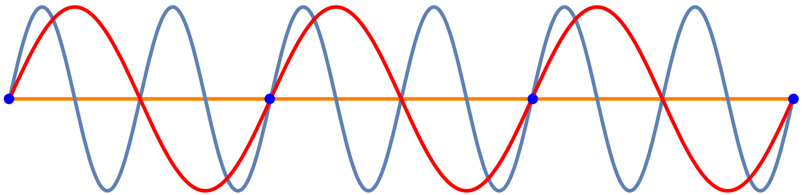

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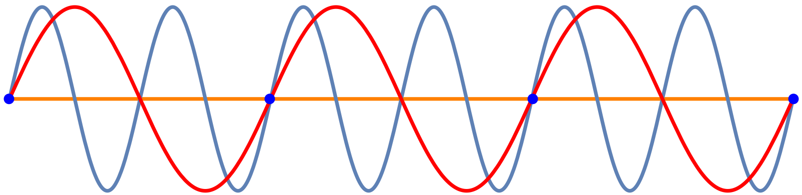

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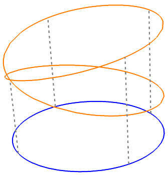
- Get "intermediate modes" corresponding to $1/2, 3/2, 5/2$, etc.

Taking a Limit

- The last frame corresponds to looking at a "twofold cover" of the circle:



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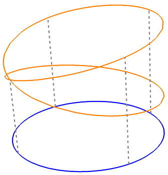

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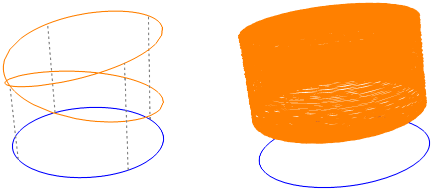

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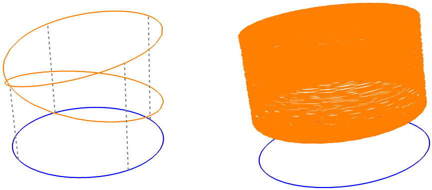

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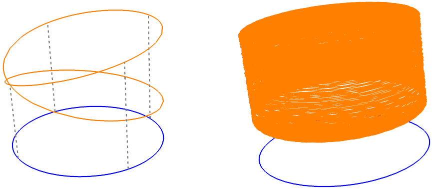

- Look at the "limit" of 2-fold, 3-fold, etc. covers:
- Get an abstract object whose modes are all positive *rational* numbers—all "levels" n
- This object is called $\mathbb{Q}\backslash\mathbb{A}$.

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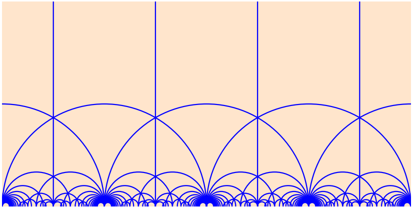

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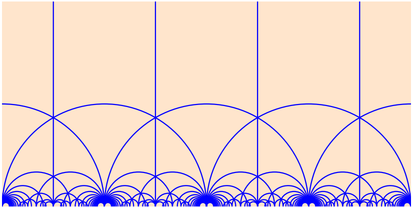

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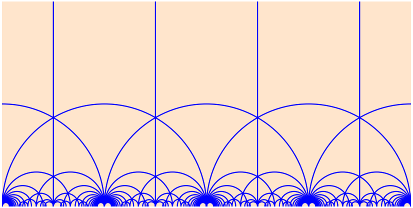

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General method: "reductive group" $\mapsto$ correct type of object

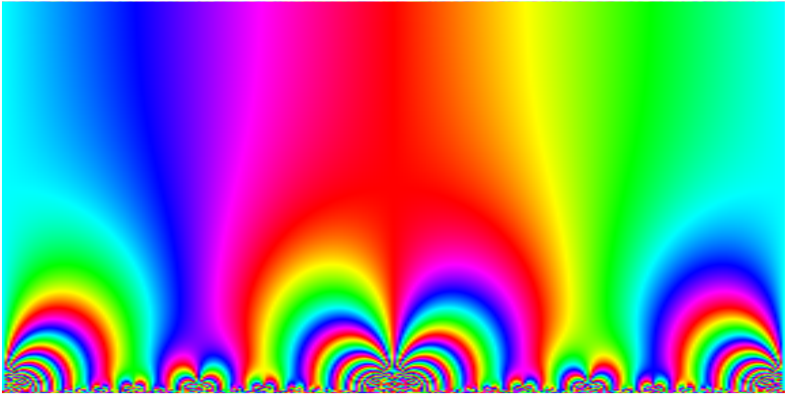

Figure: $\Delta(z)$, hue is argument

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  - First construction of optimal expander graphs $\rightarrow$ Ramanujan bound $\rightarrow$ Weil bound in number theory