Automorphic Representations and "Golden" Quantum Logic Gates

Rahul Dalal (Joint w/Shai Evra and Ori Parzanchevski)

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Note on technical details

- Anything in gray is a technical detail not relevant to this particular topic
- Anything in orange we will only explain intuitively and imprecisely due to time constraints.

Outline

- Quantum Computing Motivation
- Results/Summary of Argument
- Argument step details

Draft available at: `https://www.mat.univie.ac.at/~rdalal/GoldenGatesDraft.pdf`

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- Aut. reps. are important beyond number theory
- There are rewards for working beyond GL_N
  - maybe even a real-world application

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- In addition: approximation should be efficiently computable.

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- $n = 3$: explicit matrices can be computed from [MSG12]

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Rest of the talk: steps 1-4 in more detail

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  - sufficient: easy way to compute distance between vertices

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Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$ acts transitively on infinite $(p+1)$-regular tree

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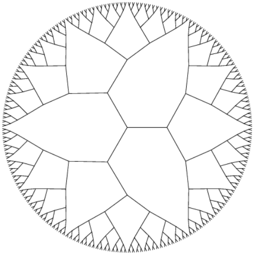

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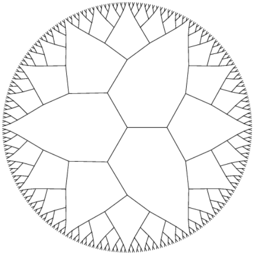

Expander Graphs Idea: $\mathrm{SL}_2 \mapsto$ compact gp. allows simple action

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Upshot: $U^{E,H}(\mathbb{Z}[1/p])$ generalizes to matrices with entries in $\mathbb{Z}[\sqrt{-d}, 1/p]$ preserving H for p inert in $\mathcal{O}_E$, $-d \equiv 3 \pmod 4$.

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- Then: Γ acts on (a $U^{E,H}(\mathbb{Q}_p)$ orbit of) vertices of the Bruhat-Tits Building $\mathcal{B}$ for $U^{E,H}(\mathbb{Q}_p)$.
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- In our case: Forget structure $\implies$ $\mathcal{B}$ a graph w/ Γ action

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Key Limitation: 1 is rarely satisfied ([MSG12]: finitely many examples with rank > 4, none with rank > 8)

- future work: find all examples with rank 4

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- Output: "$U^{E,H}(\mathbb{A}^\infty)$"-action on $L^2(U(2^n))$ understandable through information about the set $\mathcal{AR}(U^{E,H})$.

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$\implies$ whenever $f \in \pi^{K^\infty}$, $\mathbf{1}_{K^\infty S^{(\ell)} K^\infty} \star f = \mathbf{1}_{S^{(\ell)}} \star f$

p-Matrix Coefficient Decay

Updated Goal: Control $\|\mathbf{1}_{S(\ell)} \star \mathbf{1}_{B_\delta}\|_2^2$ by bounding projections of $\mathbf{1}_{B_\delta}$ onto $\pi \in \mathcal{AR}(U^{E,H})$ where $1_{S(\ell)}\star$ acts with large eigenvalues.

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- p-matrix coefficients: ways to embed π^{K^∞} into functions on $U^{E,H}(\mathbb{Q}_p)/K_p \approx \mathcal{B}$ consistently w/ Hecke-action

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Updated Goal: Control $\|\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta}\|_2^2$ by bounding projections of $\mathbf{1}_{B_\delta}$ onto $\pi \in \mathcal{AR}(U^{E,H})$ where $1_{S^{(\ell)}}\star$ acts with large eigenvalues.

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Interpretation: most of $\mathbf{1}_{\widetilde{B}_\delta}$ avoids violations of Ramanujan

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Upshot: rewrite bound in terms of Arthur-SL_2 instead of $\sigma(\pi, p)$.

Computing $a(\delta, \pi)$

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- $\mathbf{1}_{\widetilde{B}_\delta}$: slight modification of $\mathbf{1}_{B_\delta}$ for computational simplicity

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Final Step: plug in formulas for $d_\square(\lambda_\infty), a(\lambda_\infty,\delta)$ and sum!

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Given reductive matrix group $G/\mathbb{Q}_p$, there is an associated contractible simplicial complex $\mathcal{B}$ with a simplicial $G(\mathbb{Q}_p)$-action.

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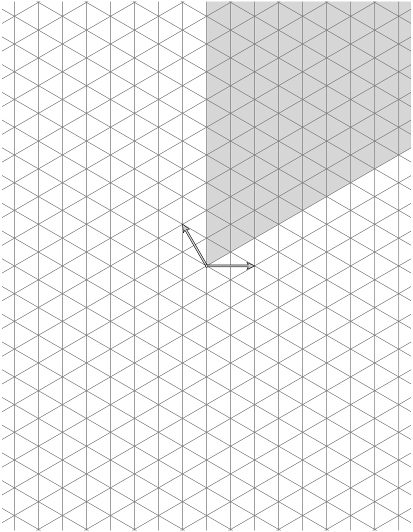
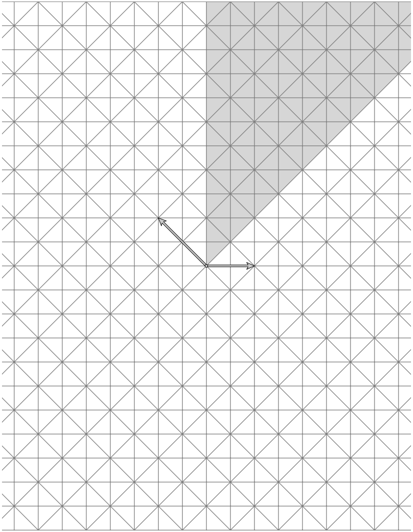
- If K is a maximal compact special subgroup, $G(\mathbb{Q}_p)/K$ embeds as a subset of the vertices of $\mathcal{B}$.
  - Consistent with $G(\mathbb{Q}_p)$-action
  - K is the stabilizer of fixed vertex v_0.
  - $\mathrm{GL}_2/\mathbb{Q}_p$: $G(\mathbb{Q}_p)/K$ is the vertices of the tree

Example Apartments





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- Good enough for rank-$4, 8$ after combinatorial casework on computer

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- James Arthur, *The L^2-Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), no. 2, 257–290. MR 1001841
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